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THE STEADY MOTIONS OF A GYROSTAT SUSPENDED ON A ROD IN A CENTRAL GRAVITATIONAL FIELD†

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The problem of the existence, stability and bifurcation of the steady motions of two bodies in an orbital tethered system, when one of the bodies is a symmetrical satellite with a rotor on the axis of symmetry, is considered. One-parameter families of steady motions are indicated, and their stability and bifurcations are investigated. The conditions which relate the parameters of the system for which stabilization of the families obtained is possible using a rotating rotor are obtained.

A considerable number of models of systems of several tethered bodies in a central Newtonian gravitational field (point masses on a rod or connected by a weightless thread, a point mass and a rigid body, a flexible heavy thread, and a system of several rigid bodies with different forms of coupling) have been proposed in the literature, as well as models of the forces acting on such a system (aerodynamic, magnetic, light pressure forces and their combinations). Restricted and unrestricted formulations of the problem have been investigated (see, for example, the monographs [1–11]). For the system considered below, which is a special case of a more general system [11], the additional symmetry of the problem leads to the existence of steady motions, which are impossible in the general case.

1. FORMULATION OF THE PROBLEM

Consider a mechanical system, consisting of a pair of rigid bodies, connected by a massless absolutely solid rod by means of two spherical hinges, in a central gravitational field. We will assume that one of the bodies moves uniformly in a circular Kepler orbit, unperturbed by the motion of the other body, which is dynamically symmetrical and carries a rotor which rotates around its axis of symmetry with an angular velocity that is constant with respect to this body.

We will assume that one of the end points of the rod – point A – moves in a circular Kepler orbit of radius R around an attracting centre N. Suppose $AX_\alpha X_\beta X_\gamma$ is an orbital system of coordinates, the unit vectors of which $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ are directed along the tangent to the orbit, along the normal to the orbital plane and along the radius vector **NA** ($\mathbf{NA} = R\boldsymbol{\gamma}$), respectively, while $\dot{\psi}$ is the modulus of the orbital angular velocity. Suppose the second end B of the rod of length l is fixed in a gyrostat, $\mathbf{AB} = l\boldsymbol{\rho}$, where $\boldsymbol{\rho}$ is also a unit vector. Suppose the centre of mass of the gyrostat G lies on its axis of symmetry at a distance a from the suspension point. Then, the radius vector $\mathbf{r}$ of the centre of mass in absolute space can be represented in the form $\mathbf{r} = R\boldsymbol{\gamma} + l\boldsymbol{\rho} + a\mathbf{s}$, where $\mathbf{s}$ is the unit direction vector of the axis of symmetry, fixed in the gyrostat. We will denote by Ω the value of the angular velocity of the rotor relative to the gyrostat, by J its axial moment of inertia, by $K = J\Omega$ the value of the natural angular momentum of the rotor.

We will introduce a system of coordinates $Gx_1x_2x_3$, connected with the gyrostat, with axes which coincide with the principal axes of inertia of the gyrostat. Henceforth, all vector quantities will be

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projected onto this system of coordinates. When $l + a \ll R$, we can use the following approximate expressions for the gravitational potential

$$U = -\frac{fmM}{R}\left(1 - 2\left(\gamma + \frac{l\rho + as}{2R}, \frac{l\rho + as}{2R}\right) + \frac{3}{2}\left(\gamma, \frac{l\rho + as}{R}\right)^2\right) -$$

$$-\frac{fm}{2R^3}(2A + C - 3(I\gamma, \gamma)); \quad I = \operatorname{diag}(A, A, C)$$

where I is the inertia tensor of the body. We will assume that $C \neq A$.

The equations of motion of the gyrostat have the form

$$m\dot{\mathbf{v}} = -m(\psi\dot{\boldsymbol{\beta}} \times \mathbf{r}) - m\boldsymbol{\omega} \times (\mathbf{v} + \psi\boldsymbol{\beta} \times \mathbf{r}) + \partial U/\partial \mathbf{r} - T\boldsymbol{\rho}$$

$$\dot{\mathbf{K}}_g = -\boldsymbol{\omega} \times \mathbf{K}_g - aT\mathbf{s} \times \boldsymbol{\rho} + \mathbf{r} \times \partial U/\partial \mathbf{r}$$

$$\dot{\boldsymbol{\beta}} = -\boldsymbol{\omega}^0 \times \boldsymbol{\beta}, \quad \dot{\boldsymbol{\gamma}} = -\boldsymbol{\omega} \times \boldsymbol{\gamma} + \psi\boldsymbol{\beta} \times \boldsymbol{\gamma}$$

$$l\dot{\boldsymbol{\rho}} = -\boldsymbol{\omega}^0 \times (l\boldsymbol{\rho} + a\mathbf{s}) + \mathbf{v}$$

where

$$\mathbf{K}_g = I\boldsymbol{\omega} + \mathbf{K}, \quad \boldsymbol{\omega} = \boldsymbol{\omega}^0 + \psi\boldsymbol{\beta}, \quad \mathbf{K} = (0, 0, K)$$

Here $\mathbf{v}$ is the velocity of the centre of mass of the gyrostat, $\boldsymbol{\omega}^0$ is the natural angular velocity of the gyrostat, $\mathbf{K}_g$ is its angular momentum, $\mathbf{K}$ is the angular momentum of the rotor and T is the reaction of the rod.

The equations of motion allow of the following five first integrals

$$\frac{1}{2}m\mathbf{v}^2 + \frac{1}{2}(I\boldsymbol{\omega}^0, \boldsymbol{\omega}^0) - \frac{1}{2}m\psi^2(\boldsymbol{\beta} \times \mathbf{r})^2 - \frac{1}{2}\psi^2(I\boldsymbol{\beta}, \boldsymbol{\beta}) - (\mathbf{K}, \boldsymbol{\beta})\psi + U = h$$

$$(\mathbf{K}_g, \mathbf{s}) = k, \quad F_1 = \boldsymbol{\beta}^2 = 1, \quad F_2 = \boldsymbol{\gamma}^2 = 1, \quad F_3 = (\boldsymbol{\gamma}, \boldsymbol{\beta}) = 0$$

These integrals express the generalized law of conservation of energy, the law of conservation of the projection of the angular momentum onto the x_3 axis (this integral only exists in the case of a symmetrical gyrostat), and also the uniqueness and orthogonality of the vectors $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$. It is also assumed that the condition for the rod to be undeformable

$$F_4 = \boldsymbol{\rho}^2 = 1$$

is satisfied.

2. TRIVIAL STEADY MOTIONS AND THEIR STABILITY

The steady motions of the system correspond to the critical points of the function

$$W_\pi = -\frac{1}{2}((I\boldsymbol{\beta}, \boldsymbol{\beta}) - m(\boldsymbol{\beta}, l\boldsymbol{\rho} + a\mathbf{s})^2) - (\mathbf{K}, \boldsymbol{\beta})\psi^{-1} +$$

$$+\frac{3}{2}((I\boldsymbol{\gamma}, \boldsymbol{\gamma}) - m(\boldsymbol{\gamma}, l\boldsymbol{\rho} + a\mathbf{s})^2) + \frac{(C\psi\beta_3 - k)^2}{2C\psi^2} +$$

$$+ 3\lambda(\boldsymbol{\gamma}, \boldsymbol{\beta}) + \frac{1}{2}\nu(\boldsymbol{\beta}^2 - 1) - \frac{3}{2}\sigma(\boldsymbol{\gamma}^2 - 1) + \frac{1}{2}\chi ml(\boldsymbol{\rho}^2 - 1)$$

where λ, ν, σ and χ are undetermined Lagrange multipliers. The system of equations for finding the critical points of the function W_π allows of the following one-parameter families of solutions

$$S_1\colon \boldsymbol{\beta} = (0,0,-\kappa_2),\quad \boldsymbol{\rho} = (0,0,-\kappa_1),\quad \gamma_3 = 0,\quad \gamma_1^2+\gamma_2^2 = 1$$

$$\sigma = A,\quad \lambda = 0,\quad \chi = -\kappa_1(l\kappa_1+a)$$

$$\nu = -m(l\kappa_1+a)^2-(K+k)\psi^{-1}\kappa_2$$

$$S_2\colon \boldsymbol{\beta} = (0,0,-\kappa),\quad \rho_3 = a/l,\quad \rho_1^2+\rho_2^2 = 1-a^2/l^2$$

$$\gamma_3 = 0,\quad \gamma_1^2+\gamma_2^2 = 1,\quad \gamma_1\rho_1+\gamma_2\rho_2 = 0$$

$$\sigma = A,\quad \lambda = 0,\quad \chi = 0,\quad \nu = -(K+k)\psi^{-1}\kappa,\quad \kappa = \kappa_{1,2} = \pm 1$$

The solution S_1 exists for any values of the parameters, and geometrically it indicates that the points A, B and G lie on one line, collinear with the vector $\boldsymbol{\beta}$ (i.e. orthogonal to the orbital plane of the point A).

The solution S_2 only exists when $a < l$, which means that the centre of mass of the gyrostat moves in the same circular orbit as the point A, its axis of symmetry is orthogonal to the orbital plane of the point A, and the rod is stress-free. In both cases the gyrostat rotates uniformly around its axis of symmetry with an arbitrary angular velocity.

An investigation of the stability of the steady motions defined by the solutions S_1 and S_2, with respect to the second variation of the function W_π on the linear manifold $\delta\mathbf{F} = (\delta F_1, \delta F_2, \delta F_3, \delta F_4) = 0$ gives the following results.

Consider the solution S_1. We introduce the dimensionless parameters

$$c = l\kappa_1+a,\quad p = \frac{l\kappa_1+a}{l},\quad q = \frac{mcl}{A},\quad x = -1-pq-\frac{\kappa_2}{A\psi}(K+k)$$

$$r = \frac{C}{A},\quad M_0 = r-1-pq-\frac{5pq+q}{p-3},\quad M = -2q+3M_0$$

We put $\kappa_1 = -1$ and suppose $M_0 < 0$. On the basis of the general theory (see, for example, [12]) the degree of instability of the solution S_1 can be represented in the form of the following table

	$x > -M-q$	$x \in (q, -M-q)$	$x < q$
$p > 3$	0	1	2
$p \in (0, 3)$	1	2	3
$p < 0$	2	3	4

If $M_0 > 0$, then q and $-M-q$ change places.

Consider the case $\kappa_1 = 1$. If $M_0 < 0$, the degree of instability depends on the parameters as follows:

$x > -M+q$	$x \in (-q, -M+q)$	$x < -q$
2	3	4

If $M_0 > 0$, $-q$ and $-M+q$ change places.

Hence, a change occurs in the degree of instability at the nodes of the table and, consequently, bifurcations of the solution S_1. Note that the angular momentum of the rotor is expressed linearly in terms of x and κ_2, which enables us to rewrite the inequalities in K or Ω easily, taking the sign of κ_2 into account.

The degree of instability of the solution S_2 is distributed as follows. Suppose

$$x = -1-\frac{\kappa}{A}(K+k)\psi^{-1},\quad p^2 = 1-\frac{a^2}{l^2},\quad r = \frac{C}{A}$$

Then, as before, we have

	$x < -3(r-1)$	$x \in (-3(r-1), 0)$	$x > 0$
$C > A$	3	2	1

If $C < 1$, then 0 and $-3(r-1)$ change places.

It can be seen that when $x = 0$ and $x = -3(r-1)$ a change occurs in the degree of instability and, consequently, branching of the solutions occurs. Note that, as in the case of the first family, the degree of instability depends on the direction of rotation of the rotor. Taking the expression for x into account, as in the case of the first family we can rewrite the conditions of stability of the solution S_2 in the form of inequalities in the angular momentum K or the angular velocity of the rotor Ω.

3. NON-TRIVIAL STEADY MOTIONS

The system for determining the critical points of the function W_π can be rewritten in the following (dimensionless) form

$$\begin{Vmatrix} 1-\sigma & \lambda & Ic_\gamma \\ 3\lambda & \nu-1 & Ic_\beta \\ -3\alpha c_\gamma & \alpha c_\beta & \chi \end{Vmatrix} \begin{Vmatrix} \gamma_i \\ \beta_i \\ \alpha\rho_i \end{Vmatrix} = 0, \quad i = 1,2 \tag{3.1}$$

$$(r-\sigma)\gamma_3 + \lambda\beta_3 - Ic_\gamma(\alpha\rho_3 - 1) = 0$$

$$3\lambda\gamma_3 + \nu\beta_3 + Ic_\beta(\alpha\rho_3 - 1) = x$$

$$-3\alpha c_\gamma\gamma_3 + \alpha c_\beta\beta_3 + \chi(\alpha\rho_3 - 1) = -\chi$$

$$\boldsymbol{\gamma}^2 = 1, \quad \boldsymbol{\beta}^2 = 1, \quad \boldsymbol{\rho}^2 = 1 \quad (\boldsymbol{\gamma}, \boldsymbol{\beta}) = 0$$

$$c_\gamma = \alpha\gamma_1\rho_1 + \alpha\gamma_2\rho_2 + \gamma_3(\alpha\rho_3 - 1)$$

$$c_\beta = \alpha\beta_1\rho_1 + \alpha\beta_2\rho_2 + \beta_3(\alpha\rho_3 - 1)$$

Here we keep the old notation

$$\sigma = \frac{\sigma}{A}, \quad \nu = \frac{\nu}{A}, \quad \lambda = \frac{\lambda}{A}, \quad \chi = \frac{\chi}{a}, \quad I = \frac{ma^2}{A}, \quad \alpha = \frac{l}{a}, \quad x = \frac{(K+k)\dot{\psi}^{-1}}{A}$$

as the new dimensionless variables.

It can be seen that if $\gamma_{1,2}$, $\beta_{1,2}$, $\rho_{1,2}$ satisfy Eqs (3.1) and at least one of them is not equal to zero, the rank of the matrix

$$D = \begin{Vmatrix} 1-\sigma & \lambda & Ic_\gamma \\ 3\lambda & \nu-1 & Ic_\beta \\ -3\alpha c_\gamma & \alpha c_\beta & \chi \end{Vmatrix}$$

must be less than three. Hence, we can consider the case when rank$D$ is equal to 0, 1, 2 and hence find the steady solutions. We denote the rows of the matrix D by d_1, d_2 and d_3.

We will consider the case when rank$D \le 1$.

Suppose $d_1 = 0$ and $d_2 = 0$. Then $\sigma = 1$, $\lambda = 0$, $\nu = 1$, $c_\gamma = 0$, $c_\beta = 0$. The system of equations takes the form

$$\chi\rho_1 = 0, \quad \chi\rho_2 = 0 \quad (r-1)\gamma_3 = 0, \quad \beta_3 = x, \quad \chi(\alpha\rho_3 - 1) = -\chi$$

$$\boldsymbol{\gamma}^2 = 1, \quad \boldsymbol{\beta}^2 = 1, \quad \boldsymbol{\rho}^2 = 1 \quad (\boldsymbol{\gamma}, \boldsymbol{\beta}) = 0$$

Note that, when analysing it, it is necessary to take the relations $c_\gamma = 0$, $c_\beta = 0$ into account. If $\chi \neq 0$, we obtain $\boldsymbol{\rho} = 0$, which contradicts the condition $\boldsymbol{\rho}^2 = 1$. If $\chi = 0$, we obtain the solution

$$S_3: \sigma = 1, \quad \lambda = 0, \quad \nu = 1, \quad \chi = 0, \quad \gamma_1\rho_1 + \gamma_2\rho_2 = 0$$

$$\gamma_1^2 + \gamma_2^2 = 1, \quad \gamma_3 = 0, \quad \beta_1\rho_1 + \beta_2\rho_2 + x(\alpha\rho_3 - 1) = 0$$

$$\beta_1^2 + \beta_2^2 = 1 - x^2, \quad \beta_3 = x, \quad \beta_1\gamma_1 + \beta_2\gamma_2 = 0, \quad \boldsymbol{\rho}^2 = 1$$

Geometrically, the solution S_3 denotes that the centre of mass of the gyrostat moves in the same circular orbit as the point A, while its axis of symmetry is inclined at a certain angle in the plane tangential to the orbit at the point G.

It can be seen that when $x = \pm 1$ this solution is identical with the solution S_2. Hence, we have obtained one of the solutions which branches from S_2.

We will determine the limits of the parameters of the system for which the solution S_3 exists. It can be shown that the coordinate ρ_3 can be expressed as follows:

$$\rho_3 = \frac{x^2}{\alpha} \pm \sqrt{(1 - x^2)(\alpha^2 - x^2)}$$

Taking into account the requirement that the radicand must be positive and that the inequalities $-1 \leq \beta_3 \leq 1$, $-1 \leq \rho_3 \leq 1$ must be satisfied, we obtain the following limits that are imposed on the parameters x and α

$$x^2 \leq 1, \quad x^2 - \alpha^2 \leq 0$$

The stability of the solution S_3 depends on the parameter $R = (C - A)/(ma^2)$, namely: if $R < 0$, the degree of instability of the solution S_3 is equal to 1 (the solution is unstable), and if $R > 0$, the degree of instability is equal to 2 (gyroscopic stabilization is possible). When $x \neq 1$ bifurcation does not occur.

We can similarly consider other cases when at least one of the rows of the matrix D is zero and rank$D = 1$. In this case we obtain two other families of steady motions (S_4 and S_5)

$$S_4: \gamma_1^2 + \gamma_2^2 = 1, \quad \gamma_3 = 0, \quad \boldsymbol{\beta} = -\kappa\boldsymbol{\rho}, \quad \beta_3 = \frac{x - \kappa I\alpha}{1 + I}$$

$$\sigma = 0, \quad \lambda = 0, \quad \nu = 1 - I\alpha\frac{\kappa x + \alpha}{1 + I}, \quad \chi = -\frac{\kappa x + \alpha}{1 + I}$$

(when $x = \kappa/\alpha \pm (1 + I)$ this solution is identical with the solution S_1; geometrically the solution S_4 denotes that the rod is perpendicular to the orbital plane of the point A, while the axis of symmetry of the gyrostat is inclined in a plane tangential to the orbit at the point A)

$$S_5: \gamma_1^2 + \gamma_2^2 = \frac{x^2}{(4 - 3r)^2}, \quad \gamma_3 = \kappa\left(1 - \frac{x^2}{(4 - 3r)^2}\right)^{1/2}$$

$$\beta_1^2 + \beta_2^2 = 1 - \frac{x^2}{(4 - 3r)^2}, \quad \beta_3 = \frac{x}{4 - 3r}$$

$$\rho_1^2 + \rho_2^2 = 1 - \frac{1}{\alpha^2}, \quad \rho_3 = \frac{1}{\alpha}, \quad 9\mu^2 = \frac{(4 - 3r)^2}{x^2} - 1$$

$$\lambda = -\frac{3\mu(r - 1)}{1 + 9\mu^2}, \quad \chi = 0, \quad \nu = 1 - \frac{3(r - 1)}{1 + 9\mu^2}, \quad \sigma = 1 + \frac{9\mu^2(r - 1)}{1 + 9\mu^2}$$

When $x^2 = (4 - 3r)^2$ this solution becomes the solution S_2. The solution S_5 denotes that the centre of mass of the gyrostat moves in the same circular orbit as the point A, and its axis of symmetry is inclined in a plane passing through the radius vector of the point A and perpendicular to the orbital plane.

4. CONCLUSION

Hence, we have obtained all the solutions which branch from the solution S_2, and one of the solutions which branches from S_1. The case when all three rows of the matrix D are proportional, and also the case when rank$D = 2$, lead to a complication of the calculations, and their investigation is not so trivial.

We will briefly formulate the main results of this paper: we have obtained the two simplest families of steady motions and the conditions for them to exist, we have investigated their stability with respect to the variable characterizing the deviation from these families, and we have found the dependence of the degree of instability on the geometrical and inertial parameters, i.e. the impossibility of stabilizing the solutions by a rotating rotor when the conditions imposed on these parameters break down, and we have found bifurcation of the steady motions. We have obtained all the solutions which branch from S_2, we have investigated the stability of one of them, we have noted that there is no bifurcation when the parameters change, and we have also obtained one of the solutions which branch off from S_1. We add that cases with a non-zero even degree of instability require additional investigation.

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